The Mystery of the Magic Square: Quantum Communication Games and What They Tell Us

- A *magic square* is a $3 \times 3$ matrix with entries in $\{0, 1\}$ such that the sum of each row is even and the sum of each column is odd.

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & \text{?} \\
\end{array}
\]

Impossible!
The Game

- Alice’s input is row $x \in \{1, 2, 3\}$.  
  Bob’s input is column $y \in \{1, 2, 3\}$.

- Alice outputs $a$, corresponding to row $x$ of a magic square.  
  Bob outputs $b$, corresponding to column $y$ of a magic square.

Intersection of Alice’s answer and Bob’s answer must agree.

$$
\begin{array}{|c|c|}
\hline
0 & \phantom{1} \\
\hline
1 & \phantom{1} \\
\hline
1 & 1 \\
\hline
\end{array}
$$
The optimal classical strategy will allow classical players to win 8 out of 9 games, on average (assuming random inputs for Alice and Bob).

So optimal \( \text{prob}_{C}(\text{win}) = \frac{8}{9} \)
Quantum Players

In the quantum winning strategy, Alice and Bob share the entangled state

$$|\psi\rangle = \frac{1}{2} |0011\rangle - \frac{1}{2} |0110\rangle - \frac{1}{2} |1001\rangle + \frac{1}{2} |1100\rangle$$

Alice and Bob apply unitary transformations $A_x$ and $B_y$, respectively, according to the following matrices:

$$A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix}, \quad A_2 = \frac{1}{2} \begin{bmatrix} -i & 1 & 1 & i \\ i & 1 & -1 & -i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{bmatrix}, \quad A_3 = \frac{1}{2} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

$$B_1 = \frac{1}{2} \begin{bmatrix} -i & -i & 1 & 1 \\ -i & -i & 1 & 1 \\ 1 & 1 & -i & -i \\ 1 & 1 & -i & -i \end{bmatrix}, \quad B_2 = \frac{1}{2} \begin{bmatrix} -1 & i & 1 & i \\ 1 & -i & 1 & -i \\ -1 & -i & 1 & -i \\ -1 & -i & 1 & -i \end{bmatrix}, \quad B_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

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For instance, if Alice has input 1 and Bob has input 2 they do:

- Alice applies $A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix}$ to her system.
- Bob applies $B_2 = \frac{1}{2} \begin{bmatrix} -1 & i & 1 & -i \\ 1 & i & 1 & -i \\ 1 -i & 1 & i \\ -1 -i & 1 & i \end{bmatrix}$, to his system.

Result is:

$$|\psi\rangle = \frac{1}{\sqrt{8}}(|0000\rangle + |0001\rangle + i|0110\rangle + i|0111\rangle - i|1000\rangle - i|1001\rangle + i|1110\rangle - i|1111\rangle)$$

Broadbent, 2008
Alice and Bob each measure their system. The result is 2 bits. The third output bit is calculated so that the sum of Alice’s bits are even, and the sum of Bob’s bits are odd.

$$|\psi\rangle = \frac{1}{\sqrt{8}}(|0000\rangle + |0001\rangle + i|0110\rangle + i|0111\rangle - i|1000\rangle - i|1001\rangle + i|1110\rangle - i|1111\rangle)$$

<table>
<thead>
<tr>
<th>Alice</th>
<th>Bob</th>
<th>Probability</th>
</tr>
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<tbody>
<tr>
<td>00</td>
<td>00</td>
<td>1/8</td>
</tr>
<tr>
<td>00</td>
<td>01</td>
<td>1/8</td>
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<tr>
<td>01</td>
<td>10</td>
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<tr>
<td>01</td>
<td>11</td>
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<td>10</td>
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<td>1/8</td>
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<tr>
<td>110</td>
<td>111</td>
<td>1/8</td>
</tr>
</tbody>
</table>

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Thus, for inputs \( x_A = 1, x_B = 2 \) Alice and Bob appear to share one of the following partial magic squares, with equal probability:

\[
\begin{array}{|c|c|c|}
\hline
0 & 0 & 0 \\
0 & & \\
1 & & \\
1 & 0 & 1 \\
0 & & \\
1 & & \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
0 & 0 & 0 \\
& 1 & \\
& 0 & \\
1 & 0 & 1 \\
& 1 & \\
& 0 & \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
0 & 1 & 1 \\
& 0 & \\
& 0 & \\
1 & 1 & 0 \\
& 1 & \\
& 0 & \\
\hline
\end{array}
\quad
\begin{array}{|c|c|c|}
\hline
0 & 1 & 1 \\
& 1 & \\
& 1 & \\
1 & 1 & 0 \\
& 1 & \\
& 1 & \\
\hline
\end{array}
\]

Broadbent, 2008
Bell Game

Alice

\[
\begin{align*}
  a &= 0 \text{ or } 1 \\
p(a=0|x) &= p(a=1|x) = 1/2
\end{align*}
\]

Bob

\[
\begin{align*}
  b &= 0 \text{ or } 1 \\
p(b=0|y) &= p(b=1|y) = 1/2
\end{align*}
\]

\[
x = 0 \text{ or } 1
\]

\[
y = 0 \text{ or } 1
\]

Winning correlation: \( a \oplus b = x \cdot y \)

- \( x, y = 00 \text{ or } 01 \text{ or } 10 \): outputs same (00 or 11)
- \( x, y = 11 \): outputs different (01 or 10)

No winning strategy!

If Alice and Bob share a random variable \( \lambda \in \{0,1\} \) and respond:

\( a = \lambda, b = \lambda \), irrespective of the input, then: \( p(\text{win}) = 3/4 \)
PR-box (nonlocal information channel)

\[ a \oplus b = x \cdot y \]

S. Popescu & D. Rohrlich
*Foundations of Physics* 24, 379, 1994

Sandu Popescu
Daniel Rohrlich
Bell's Theorem $\Rightarrow |K_c| \leq 2$, for classical ‘common cause’ correlations.
So:

$$1/4 \leq \text{prob}_c(\text{successful sim}) \leq 3/4$$

optimal $\text{prob}_c(\text{successful sim}) = 3/4$

i.e., optimal $\text{prob}_c(\text{win}) = 3/4$ for EPR game

Remarkably, if Alice and Bob are allowed to share entangled quantum states, then there is an optimal quantum strategy such that:

$$\text{prob}_Q(\text{success sim}) \approx .85$$

i.e., optimal $\text{prob}_Q(\text{win}) \approx .85$ for EPR game!

For quantum systems, Tsirelson’s inequality $\Rightarrow |K_Q| \leq 2\sqrt{2}$

So, optimal $\text{prob}_Q(\text{successful sim}) = (2\sqrt{2})/8 + 1/2 \approx .85$

For a PR-box, $K_{PR} = 4$
Why quantum mechanics?

Why QM (prob(win) ≈ .85) and not CM (prob(win) = .75)?

Why can’t we win game all the time (prob(win) = 1)?

What principle excludes the construction of PR-boxes?

No violation of ‘no signaling’ or logic in any of these cases.
We always have had a great deal of difficulty in understanding the world view that quantum mechanics represents. ... I have entertained myself always by squeezing the difficulty of quantum mechanics into a smaller and smaller place, so as to get more and more worried about that particular item. It seems to be almost ridiculous that you can squeeze it to a numerical question that one thing is bigger than another. But there you are – it is bigger.

Schematic representation of the space of ‘no signaling’ correlations:
The vertices are labelled L and NL for local and nonlocal. Bell inequalities are the facets represented by dashed lines. The region accessible to quantum correlations is Q. A general nonlocal box lies in the region P.

Nonlocal correlations are monogamous

PR-box correlations are monogamous.
So are the correlations of all ‘no signaling’ theories - except classical theories.

Alice:
- $x = 0$ or $1$
- $a = 0$ or $1$

Bob:
- $y = 0$
- $b = 0$ or $1$

Charles:
- $z = 1$
- $c = 0$ or $1$

$a \oplus b = x \cdot y$
$a \oplus c = x \cdot z$
$b \oplus c = x \cdot (y \oplus z) = x$

If not, Bob and Charles could compute $x$, the value of Alice’s input – so Alice could signal to Bob and Charles.
Monogamy of nonlocal correlations

(0, 2\sqrt{2})

'no signalling' region

classical region

quantum region

Toner and Verstraete
quant-ph/0611011
Alice:
\[ x = 0 \text{ or } 1 \]
\[ a = 0 \text{ or } 1 \]

Bob:
\[ y = 0 \]
\[ b = 0 \text{ or } 1 \]

\[ a \oplus b = x \cdot y \]
\[ a \oplus b' = x \cdot y \]

\[ b \oplus b' = x \cdot (y \oplus y') = x \]

If Bob could clone his part of a PR-box, Bob could compute \( x \) – the value of Alice’s input – so Alice could use the cloned channel to signal to Bob.
No cloning $\Rightarrow$ ‘uncontrollable’ measurement disturbance
(i.e., extracting information from an unclonable source changes the source irreversibly).
There exists a nonlocal winning strategy for the magic square game that makes use of a single NLB.

Proof: Alice and Bob each have two strategies, $A_0$ and $A_1$ for Alice and $B_0$ and $B_1$ for Bob such that:

- all strategies respect the parity condition
- Both pairs of strategies $(A_0, B_0)$ and $(A_1, B_1)$ yield a correct answer for all inputs except $x = y = 3$.
- Both pairs of strategies $(A_0, B_1)$ and $(A_1, B_0)$ yield a correct answer when $x = y = 3$.

Now, Alice and Bob each input 1 into the NLB if their input is 3 (and otherwise they input 0). They use the output of the NLB to determine which strategy to use.

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- In the quantum winning strategy, the outcomes of the players are uniformly distributed.

- By randomizing over all possible strategies $A_0, A_1, B_0, B_1$, it is possible to simulate the correlations of the Magic Square game.

- Corollary: A NLB can simulate bipartite correlations that no 2-qubit entangled state, $\alpha|00\rangle + \beta|11\rangle$, can.

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